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MTH 512, Exam-I

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$$\overline{\quad\quad} \\ 65$$

QUESTION 1. Let $T : R^3 \to R^2$ be an $\mathbb{R}$-homomorphism that is ONTO. Given $T(2,1,5) = (0,0)$, $T(1,0,2) = (3,0)$, and $T(0,1,0) = (0,5)$. Find all points in R^3, say (a, b, c), such that $T(a,b,c) = (6,-5)$

QUESTION 2. (a) Let $1 \leq n < \infty$. Now, let $a_1, a_2, ..., a_n, b_1, ..., b_n \in R$. Prove that

$$|\sum_{i=1}^{n} a_i b_i| \leq \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}$$

(b) Prove that

$$(\sum_{i=1}^{n} a_i)^2 \leq n \sum_{i=1}^{n} a_i^2$$

QUESTION 3. (a) Let $x, y \in V$, where V is a real inner product vector space. Prove that

$$|x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2)$$

(b) Let V be a normed finite dimensional vector space and $T : V \to V$ be a linear transformation such that $||T(v)|| \leq 3||v||$. Prove that $T - \sqrt{11}I$ is an invertible linear transformation from V ONTO V.

(c) Let $T : R^2 \to R^2$ be a linear transformation such that $T(3,5) = (6,10)$, and $T - 5I : R^2 \to R^2$ is a non-invertible linear transformation. Prove that T is invertible. [Hint: $T(3,5) = 2(3,5)$]

QUESTION 4. (i) Give me an example of a normed vector space V where $||u+w|| = 2||u||$, for some $u, w \in V$, but $||u|| \neq ||w||$.

(ii) Given $T : R^3 \to R^2$ is a linear transformation, B is a basis for R^3 and $C = \{(1,1), (-1,1)\}$ is a basis for R^2. Assume that $[T]_{B,C} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$. Find the standard matrix M_T and $Ker(T)$.

(iii) We know that $< f_1, f_2 >= \int_0^1 f_1 f_2 \, dx$ for every $f_1, f_2 \in P_3$ is an inner product on P_3. Let $D = span\{x^2, x\}$. Find $D^\perp$.

(iv) Let $< \, , \, >$ be the normal dot on R^4. Given $D = span\{(1,1,1,1), (-2,-3,-1,-2)\}$ and $dim(D) = 2$. Find the point $d \in D$ that is the nearest to the point $Q = (16,-4,4,-4) \notin D$, i.e., find the point d in D such that $|Q - d|$ is minimum. [Hint: OPTIONAL, maybe the kill below method gives you an orthogonal basis for D]

QUESTION 5. Let $< \, , \, >$ be the normal dot on R^n and R^m. Given $T : R^n \to R^m$ is a linear transformation. Let $\{e_1, ..., e_n\}$ be the standard basis of R^n and $\{b_1, ..., b_m\}$ be the standard basis of R^m.

(i) Prove that there are unique m points in R^n, say $q_1, ..., q_m \in R^n$ such that

$$T(v) = (< q_1, v >, < q_2, v >, ..., < q_m, v >)$$

for every $v \in R^n$. [Hint: just translate some familiar math!]

(ii) Prove that $\dim(Ker(T^*)) = dim((Range(T))^\perp)$ [Hint : Let L be the co-linear of T^*, show that $Ker(L) = (Range(T))^\perp$ (maybe (i) is useful). Hence the translation of Ker(L) to the language of $(R^m)^*$ is the $Ker(T^*)$. Thus $dim(Ker(L)) = dim(Ker(T^*)) = dim((Range(T))^\perp)$]

(iii) Assume $R^n = R^2$ and $R^m = R^3$ and $Range(T) = span\{q_1 = (1,1,1), \ q_2 = (-1,-1,0)\}$ (note that q_1, q_2 are independent). Find $Ker(T^*)$.

Faculty information

Exam 1. $\frac{54}{65}$ v. good Imane SAHEL

Question 1: let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be an $\mathbb{R}$-homomorphism that is onto

Find all points in $\mathbb{R}^3$, say (a,b,c) st. $T(a,b,c) = (6,-5)$.

we have $T(2,1,5) = (0,0)$

$T(1,0,2) = (3,0)$

$T(0,1,0) = (0,5)$

the three points $(2,1,5)$, $(1,0,2)$, $(0,1,0)$ are linearly independent. so they form a basis for $\mathbb{R}^3$. and any vector in $\mathbb{R}^3$ is a unique linear combination of these three vectors. moreover the image of any V in $\mathbb{R}^3$ is determined by the image of these 3 vectors

let $V \in \mathbb{R}^3$, then $T(V) = c_1 T(2,1,5) + c_2 T(1,0,2) + c_3 T(0,1,0)$.

$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

the solution set to the above system $= \{(t, 2, -1), t \in \mathbb{R}\}$

thus $V = t(2,1,5) + 2(1,0,2) - (0,1,0) = \{t(2,1,5) + (2,-1,4), t \in \mathbb{R}\}$

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Question 2: a) let $1 \leq n \leq \infty$. Now let $a_1 \dots a_n, b_1 \dots b_n \in \mathbb{R}$. Prove that

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}$$

we know that $\mathbb{R}^n$ is a vector space over $\mathbb{R}$

and $\langle (a_1 \dots a_n), (b_1 \dots b_n) \rangle = a_1 b_1 + a_2 b_2 \dots a_n b_n = \sum_{i=1}^{n} a_i b_i$

is an inner product space.

so we can apply schwarz inequality.

$$|\langle (a_1 \dots a_n), (b_1 \dots b_n) \rangle| \leq \sqrt{\langle (a_1 \dots a_n), (a_1 \dots a_n) \rangle} \sqrt{\langle (b_1 \dots b_n), (b_1 \dots b_n) \rangle}$$

so

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \sqrt{\sum_{i=1}^{n} a_i^2} \sqrt{\sum_{i=1}^{n} b_i^2} = \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}$$

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b). Prove that $\left|\sum_{i=1}^{n} a_i\right|^2 \leq n \sum_{i=1}^{n} a_i^2$

let $b_i = a_i$ in ①

then we get $\sum_{i=1}^{n} a_i a_i = \sum_{i=1}^{n} a_i^2 \leq \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} a_i^2}$

$$= \sqrt{\left(\sum_{i=1}^{n} a_i^2\right)^2}$$

$$= \sum_{i=1}^{n} a_i^2$$

1/5

that is $\sum_{i=1}^{n} a_i^2 \leq \sum_{i=1}^{n} a_i^2$

and since $n \geq 1$ then $\leq n \sum_{i=1}^{n} a_i^2$

Just use a and let $b = (1, 1, \dots, 1)$

This does not mean $\left(\sum a_i\right)^2 \leq$!

Question 3: let $x, y \in V$, where V is a real inner product vector space.

a]. Prove that. $|x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|)^2$

$|x+y|^2 + |x-y|^2 = \langle x+y, x+y \rangle + \langle x-y, x-y \rangle$

$= \langle x,x \rangle + \langle x,y \rangle + \langle y,x \rangle + \langle y,y \rangle + \langle x,x \rangle - \langle x,y \rangle - \langle y,x \rangle + \langle y,y \rangle$

5 $= |x|^2 + 2\cancel{\langle x,y \rangle} + |y|^2 + |x|^2 - 2\cancel{\langle x,y \rangle} + |y|^2$

$= 2(|x|^2 + |y|^2)$

b]. let V be a normed finite dimensional vector space and $T: V \to V$ is a linear transformation s.t. $\|T(v)\| < 3\|v\|$.

Prove that $T - \sqrt{11}\,I$ is an invertible linear transformation.

→ to show that $T - \sqrt{11}\,I$ is invertible, we need to show that it is a bijection, i.e. injective and surjective

but if it is injective then $\dim(\mathrm{Ker}(T - \sqrt{11}\,I)) = 0$

thus $\dim(V) = \dim(\mathrm{Range}(T - \sqrt{11}\,I)) = n < \infty$

so it must be surjective

so it suffices to show that the null space of $T - \sqrt{11}\,I$ is the trivial one

suppose not, then $\exists\, v \neq 0_V$ s.t. $(T - \sqrt{11}\,I)(v) = 0_V$

that is $\sqrt{11}$ is an Eigen value of T

hence $\exists\, v \neq 0_V$ s.t. $T(v) = \sqrt{11}\, v$

so $\|T(v)\| = \|\sqrt{11}\, v\| = |\sqrt{11}|\,\|v\|$

but $\|T(v)\| < 3\|v\| \quad \forall\, v \in V$

a contradiction so $T - \sqrt{11}\,I$ is invertible

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Good

c]: let $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T(3,5) = (6,10) = 2(3,5)$

so 2 is an Eigen value of T

$T - 5I: \mathbb{R}^2 \to \mathbb{R}^2$ is a non invertible linear transformation

means that $\exists\, v \neq 0_V$ in the null space of $T - 5I$.

so 5 is an Eigen value of T.

obviously $\dim(E_2) = 1$ and $\dim(E_5) = 1$

so $\text{Range}(T) = E_2 + E_5 = \mathbb{R}^2$

so T is surjective.

let v_1 the eigen vector corresponding to 2

v_2 the Eigen vector " to 5

let $w \in \mathbb{R}^2$ then $w = c_1 v_1 + c_2 v_2$

$T(w) = c_1 T(v_1) + c_2 T(v_2)$

$T(w) = 2c_1 v_1 + 5 c_2 v_2$

since v_1, v_2 are linearly independent then $T(w) = (0,0)$

iff $c_1 = c_2 = 0$ so $\text{Ker}(T) = \{0_V\} \Rightarrow T$ is injective

$\Rightarrow$ so it is invertible.

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Question 4:

2). $T: \mathbb{R}^3 \to \mathbb{R}^2$ is a L.T. B is a basis for $\mathbb{R}^3$

$C = \{(1,1), (-1,1)\}$ is a basis for $\mathbb{R}^2$

Assume $[T]_{B,C}$ as given. Find the standard matrix M_T

we know $[T]_{B,C} = C^{-1} M_T B^{-1}$

(assuming B is the ordered basis).

so $C[T]_{B,C} = M_T$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}_{2\times 3} = \begin{array}{c} \begin{matrix} a & b & c \end{matrix} \\ \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = M_T$$

$\to \ker(T) = \{(-b-c, b, c)\}$
$= \{b(-1, 1, 0) + c(-1, 0, 1)\}$
$= \text{span}\{(-1,1,0), (-1,0,1)\}$

$$\ker(T) = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},\ c_1, c_2 \in \mathbb{R} \right\}$$

$\ker(T) = \text{span}\{(1, 0, -1)^T, (0, 1, -1)^T\}$

4/5

3). $D^\perp = \{ f \in P_3 : \langle f, x^2 \rangle = 0 \wedge \langle f, \cancel{x} \rangle = 0 \}$ $\to$??

let $f(x) = ax^2 + bx + c$

$$\int_0^1 (ax^2 + bx + c)x^2 dx = \int_0^1 ax^4 + bx^3 + cx^2 dx = \frac{a}{5} + \frac{b}{4} + \frac{c}{3}$$

$$\int_0^1 \cancel{x}(ax^2 + bx + c) dx = \frac{a}{3} + \frac{b}{2} + c$$

so we are looking for

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & 1 \\ \frac{1}{5} & \frac{1}{4} & \frac{1}{3} \end{bmatrix}_{2\times 3} \begin{bmatrix} a \\ b \\ c \end{bmatrix}_{3\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

solving this system we get $a = 5c$, $b = \frac{-16}{3} c$, $c \in \mathbb{R}$

$D^\perp = \text{span}\{5x^2 - \frac{16}{3}x + 1\}$

6 on wrong

4/4

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4) let $v_1 = (1, 1, 1, 1)$

then $v_2 = (-2, -3, -1, -2) - \frac{-8}{18}(1, 1, 1, 1)$

4 ??

$v_2 = \left(\frac{-14}{9}, \frac{-23}{9}, \frac{-5}{9}, \frac{-14}{9}\right)$

thus $B = \{v_1, v_2\}$ is an orthogonal basis for D.

Use my hint!

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -3 & -1 & -2 \end{bmatrix} \xrightarrow{2R_1+R_2 \to R_2}$

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$ ortho gonal

D = span{(1,1,1,1), (0,-1,1,0)}

$d = \frac{\langle Q, v_1 \rangle}{|v_1|^2} v_1 + \frac{\langle Q, v_2 \rangle}{|v_2|^2} v_2$

$d = \frac{12}{4}(1, 1, 1, 1) + \frac{\frac{-176}{9}}{\frac{946}{81}}\left(\frac{-14}{9}, \frac{-23}{9}, \frac{-5}{9}, \frac{-14}{9}\right)$.

3/5

1). let $\|\cdot\| : \mathbb{R}^3 \to \mathbb{R}$

$\|(a_1, a_2, a_3)\| = \max\{|a_i| ; 1 \leq i \leq 3\}$

let $u = (-3, 0, 2)$ $\qquad \|u\| = 3$

$v = (-2, 0, 4)$ $\qquad \|v\| = 4$

$\|u + v\| = 6 = 2\|u\|$

$u + v = (-5, 0, 6)$ ✓

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Question 5: let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with standard ordered basis

then T has a matrix representation let's call it M

then $M = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) & \cdots & T(e_n) \end{bmatrix}_{m \times n}$

for $v \in \mathbb{R}^n$

$$T(v) = M_{m\times n} \, v_{n \times 1}$$

thus every row in M is a point in $\mathbb{R}^n$ there are m of the let's call them $a_1 \cdots a_m$

hence $T(v) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

a_1

a_2

5/5

$$= \left(a_{11} v_1 \cdots a_{1n} v_n ,\ a_{21} v_1 \cdots a_{2n} v_{n-1} \cdots \right)^T$$

$$= \left(\langle a_1, v \rangle , \langle a_2, v \rangle , \cdots \langle a_m, v \rangle \right)$$

2) $\dim(\mathrm{Range}(T))^{\perp} = \dim(\mathrm{codomain}) - \dim(\mathrm{Range}\, T)$
$\equiv m - \#$ of independent columns or rows

if $T: V \longrightarrow W$, n, $= m$

then $T^*: W^* \longrightarrow V^*$, m, n

$\dim(\ker(T^*)) = \dim(W^*) - \dim(\mathrm{Range}\, T^*) = m - \#$ of independent columns

since if M is the matrix representation of T then M^T is the $\sim$ $\sim$ of T^*

0/5

So $\dim(\mathrm{Range}(T)) = \dim(\mathrm{Range}\, T^*)$

$T: \mathbb{R}^{n} \longrightarrow \mathbb{R}^{m}$

$T: \mathbb{R}^{2} \longrightarrow \mathbb{R}^{3}$

Range $(T) = \mathbb{R}^2$

$$M_T = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}_{3\times 2}$$

$\dim \ker(T^*) = 1$

$T^*: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$M_{T^*} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$\ker(T^*) = \operatorname{span}\{(-1, 1, 0)\} = \operatorname{span}\{-e_1^* + e_2^*\}$

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You need to prove that $\ker(T^*) = [\text{Range}(T)]^{\perp}$

If $\dim(V) = \dim(W)$, then Maybe $V = W$ and Maybe $V \neq W$

From 5(2) $\Rightarrow$ we have $\dim(\ker(T^*)) = \dim([\text{Range}(T)]^{\perp})$

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